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Some Examples and thm.

Ex 17) Let S be a subset of an I.P.S. V. Then show that (i) $S^\perp = [L(S)]^\perp$

(ii) $L(S) \subset S^{\perp\perp}$, where $L(S)$ is Linear Span of S or generating set of S.

Solution:-

(i) Since $L(S)$ is the linear span of S

$\Rightarrow S \subset L(S)$

$\Rightarrow [L(S)]^\perp \subset S^\perp$ — (I)

{ By Prop. of complement $W_1 \subset W_2$, $W_2^\perp \subset W_1^\perp$ }

R.T.P

$S^\perp \subset [L(S)]^\perp$

Let $S = \{u_1, u_2, \ldots, u_n\}$

Then any $v \in L(S)$ can be written as

$v = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \quad \forall \alpha_i \in F$

$v = \sum_{i=1}^{n} \alpha_i u_i$ — (*)

{ $v \in L(S)$, v can be written as linear comb. of elts of S }

We know that $S^\perp = \{u \in V \,/\, \langle u, v\rangle = 0$ for all $v \in S\}$

Let $u \in S^\perp$ Then $\langle u, u_i\rangle = 0$ for all $u_i \in S$, $1 \le i \le n$.

Now $\langle u, v\rangle = \langle u, \sum \alpha_i u_i\rangle$ [Using (*)]

$= \langle u, \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n\rangle$

$= \bar{\alpha}_1 \langle u, u_1\rangle + \bar{\alpha}_2 \langle u, u_2\rangle + \cdots + \bar{\alpha}_n \langle u, u_n\rangle$

$= \sum_{i=1}^{n} \bar{\alpha}_i \langle u, u_i\rangle$ { $\because \langle u, \alpha v\rangle = \bar{\alpha}\langle u, v\rangle$ }

$= \sum \bar{\alpha}_i \cdot 0$ { $\because \langle u, u_i\rangle = 0$ }

$= 0$

$\Rightarrow \langle u, v\rangle = 0 \Rightarrow u$ is perpendicular to v

$\Rightarrow u \perp L(S)$ { $\because v \in L(S)$ }

$\Rightarrow u \in [L(S)]^\perp$ { By def. of ortho. complement }

$\therefore \quad u \in S^{\perp} \Rightarrow u \in [L(S)]^{\perp}$

$\therefore \quad S^{\perp} \subset [L(S)]^{\perp}$ — (2)

From (1) & (2) we have

$S^{\perp} = [L(S)]^{\perp}$

Similarly, we can Prove 2nd part Also.

Ex 18) Let $V(R)$ be a real inner Product space. Let $\alpha, \beta \in V$ such that $\alpha \perp \beta$. Then show that $\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$. Also Prove converse.

Solution:- $\alpha, \beta \in V$ and $V(R)$ is an I.P.S.

Now

$\|\alpha+\beta\|^2 = \langle \alpha+\beta, \alpha+\beta \rangle = \langle \alpha, \alpha+\beta \rangle + \langle \beta, \alpha+\beta \rangle$

$= \langle \alpha,\alpha \rangle + \langle \alpha,\beta \rangle + \langle \beta,\alpha \rangle + \langle \beta,\beta \rangle$

$= \|\alpha\|^2 + \langle \alpha,\beta \rangle + \overline{\langle \alpha,\beta \rangle} + \|\beta\|^2$

$= \|\alpha\|^2 + \langle \alpha,\beta \rangle + \langle \alpha,\beta \rangle + \|\beta\|^2$

$= \|\alpha\|^2 + 2\langle \alpha,\beta \rangle + \|\beta\|^2$

$\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 + 2\langle \alpha,\beta \rangle$

By Prop. of I.P.S. $\langle \alpha,\beta \rangle = \overline{\langle \beta,\alpha \rangle}$ OR $\langle \beta,\alpha \rangle = \overline{\langle \alpha,\beta \rangle}$

$\because V(R) \to$ I.P.S. $F = R$ $\therefore \overline{\langle \alpha,\beta \rangle} = \langle \alpha,\beta \rangle$

Also given $\alpha \perp \beta \Rightarrow \langle \alpha,\beta \rangle = 0$

$\therefore \|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$

Conversely:

Let $\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$ — (*)

T.P. $\alpha \perp \beta$.

We know that, Here

$\|\alpha+\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 + 2\langle \alpha,\beta \rangle$ — (**)

using (*) in (**)

$\|\alpha\|^2 + \|\beta\|^2 = \|\alpha\|^2 + \|\beta\|^2 + 2\langle \alpha,\beta \rangle$

$\Rightarrow 2\langle \alpha,\beta \rangle = 0$

$\Rightarrow \langle \alpha,\beta \rangle = 0 \Rightarrow \alpha \perp \beta$

orthonormal Set:-

Let $S = \{u_1, u_2, \ldots, u_n\}$ be a subset of I.P.S. V. Then 'S' is s.t.b. orthonormal set if it satisfy two cond:-

(i) $\langle u_i, u_i \rangle = 1$ i.e $\|u_i\| = 1$ for $1 \le i \le n$.

(ii) $\langle u_i, u_j \rangle = 0$ for $i \ne j$; $1 \le i, j \le n$.

Example:- Consider the standar basis of R^2.

$$S = \{ \underbrace{(1,0)}_{e_1}, \underbrace{(0,1)}_{e_2} \}$$

Now $\langle e_1, e_1 \rangle = \langle (1,0), (1,0) \rangle = 1 \cdot 1 + 0 \cdot 0 = 1$.

$\langle e_2, e_2 \rangle = \langle (0,1), (0,1) \rangle = 0 \cdot 0 + 1 \cdot 1 = 1$.

Also $\langle e_1, e_2 \rangle = \langle (1,0), (0,1) \rangle = 1 \cdot 0 + 0 \cdot 1 = 0$

$\therefore$ S is orthonormal set of R^2.

Thm:- Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of an I.P.S. V and $u \in V$ be arbitrary. Then the co-ordinates of u relative to the basis $\{u_i\}$ are $\langle u, u_i \rangle$ and $\|u\|^2 = \sum_{i=1}^{n} |\langle u, u_i \rangle|^2$

Proof:- Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis of I.P.S. V(F).

Then (i) $\langle u_i, u_i \rangle = 1 \quad \forall\ 1 \le i \le n$

(ii) $\langle u_i, u_j \rangle = 0 \quad ; i \ne j$

[By def. of orthonormal set]

Let $u \in V$. and $\{u_1, u_2, \ldots, u_n\}$ is a basis of V.

$$\therefore \quad u = \sum_{i=1}^{n} \alpha_i u_i \quad ; \alpha_i \in F \qquad (1)$$

By definition of co-ordinates of u relatives to basis $\{u_i\}$ are scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Now, for each i; $\langle u, u_i \rangle = \langle \sum_{j=1}^{n} \alpha_j u_j , u_i \rangle$

$= \langle \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n , u_i \rangle$

$= \alpha_1 \langle u_1, u_i \rangle + \alpha_2 \langle u_2, u_i \rangle + \cdots + \alpha_n \langle u_n, u_i \rangle$

Take $u = \sum_{j=1}^{n} \alpha_j u_j$ for different index

$= \sum_{j=1}^{n} \alpha_j \langle u_j, u_i \rangle$ for all $1 \le i \le n$.

$\langle u, u_i \rangle = \alpha_i$ for $1 \le i \le n$.

$=$ Co-ordinates of u relative to the basis $\{u_i\}$

$\left\{ \begin{array}{l} \text{When } i=j \\ \langle u_i, u_j \rangle = 1 \\ i \ne j \ \langle u_i, u_j \rangle = 0 \end{array} \right.$

Now; Remain to Prove $\|u\|^2 = \sum_{i=1}^{n} |\langle u, u_i \rangle|^2$

From ① $u = \sum_{i=1}^{n} \alpha_i u_i = \sum_{i=1}^{n} \langle u, u_i \rangle \cdot u_i$ $\left\{ \because \langle u, u_i \rangle = \alpha_i \right\}$

Now.

$\|u\|^2 = \langle u, u \rangle = \langle \sum_{i=1}^{n} \alpha_i u_i , \sum_{j=1}^{n} \alpha_j u_j \rangle$

$= \langle \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n , \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \rangle$

$= \alpha_1 \langle u_1 , \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \rangle$
$+ \alpha_2 \langle u_2 , \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \rangle$
$+ \cdots + \alpha_n \langle u_n , \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \rangle$

$= \sum_{i=1}^{n} \alpha_i \langle u_i , \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n \rangle$

$= \sum_{i=1}^{n} \alpha_i \langle u_i , \sum_{j=1}^{n} \alpha_j u_j \rangle$

$= \sum_{i=1}^{n} \alpha_i \left[\sum_{j=1}^{n} \bar{\alpha}_j \langle u_i, u_j \rangle \right]$

$= \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \bar{\alpha}_j \langle u_i, u_j \rangle = \sum_{i=1}^{n} \alpha_i \bar{\alpha}_i$ $\left\{ \begin{array}{l} \because \text{When } i=j \\ \langle u_i, u_j \rangle = 1 \end{array} \right.$

$\|u\|^2 = \sum_{i=1}^{n} |\alpha_i|^2$

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