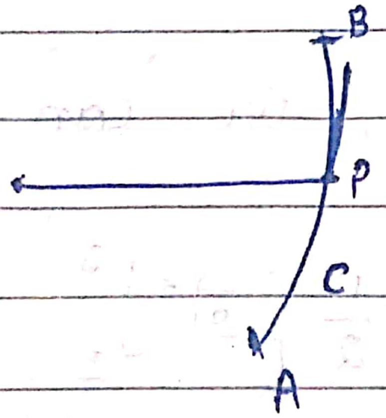
* **Line Integral** :→ An integral which is to be determined along a curve is called a line integral.



Let C be a curve with starting point as A and terminating point as B.

Let P be any point on curve C,

i.e, $P(x, y, z)$ is any point on C,

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of this point $P(x, y, z)$ on C

(x, y, z are the f^ns of variable t).

Then,

Tangent vector to the curve C at P is,

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Let s represents arc length of the curve then unit vector along the tangent to the curve C at point P is,

$$t = \frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \frac{\frac{d\vec{r}}{dt}}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}}$$

$$= \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds}$$

Let $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ be a vector point f^n and cnts along C. Then, component of $\vec{f}$ along the tangent at P is,

$$\vec{f}\cdot\hat{t} = \vec{f}\cdot\frac{d\vec{r}}{ds}$$

Clearly, $\vec{f}\cdot\hat{t}$ is a f^n of s and,

$$\int_A^B (\vec{f}\cdot\hat{t})\,ds = \int_A^B (\text{components of } \vec{f} \text{ along the tangent})\,ds$$

is known as line integral of $\vec{f}$ on C.

Imp

The line integral can be written as,

$$\int_A^B (\vec{f}\cdot\hat{t})\, ds = \int_A^B \left(\vec{f}\cdot\frac{d\vec{r}}{ds}\right) ds = \int_A^B \vec{f}\cdot d\vec{r}$$

$$= \int_C \vec{f}\cdot d\vec{r}$$

Now, $\int_C \vec{f}\cdot d\vec{r} = \int_C (f_1\hat{i} + f_2\hat{j} + f_3\hat{k})\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \int_C (f_1\,dx + f_2\,dy + f_3\,dz)$$

If the curve C is given in the parametric form $x = x(t)$, $y = y(t)$, $z = z(t)$, then

$$\int_C \vec{f}\cdot d\vec{r} = \int_C \vec{f}\cdot\frac{d\vec{r}}{dt}\,dt = \int_C \left[f_1\frac{dx}{dt} + f_2\frac{dy}{dt} + f_3\frac{dz}{dt}\right] dt$$

Que → Evaluate the integral $\int_C \vec{f}\cdot d\vec{r}$

where, $\vec{f} = xy\hat{i} + yz\hat{j} + zx\hat{k}$,
Curve C is $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$
varies from -1 to 1.

Solⁿ → We want to evaluate line integral,

$$\int_C \vec{f} \cdot d\vec{r}$$

and, we knows

$$\int_C \vec{f} \cdot d\vec{r} = \int_C \left(\vec{f} \cdot \frac{d\vec{r}}{dt}\right) dt \longrightarrow ①$$

$$\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k} \Rightarrow \boxed{x = t,\ y = t^2,\ z = t^3}$$

$$\frac{d\vec{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$$

$$\vec{f} \cdot \frac{d\vec{r}}{dt} = (xy\hat{i} + yz\hat{j} + zx\hat{k}) \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k})$$

$$= (t^3\hat{i} + t^5\hat{j} + t^4\hat{k}) \cdot (\hat{i} + 2t\hat{j} + 3t^2\hat{k})$$

$$= (t^3 + 2t^6 + 3t^6)$$

$$= t^3 + 5t^6$$

Put this value of $\vec{f} \cdot \frac{d\vec{r}}{dt}$ in ①,

$$\int_C \vec{f} \cdot d\vec{r} = \int_C (t^3 + 5t^6)\, dt$$

$$= \int_{-1}^{1} (t^3 + 5t^6)\, dt \quad \left(\because \underline{\text{Given}} \rightarrow \text{curve is Bdd b/w } -1 \text{ and } 1\right)$$

$$= \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_{-1}^{1}$$

$$= \left[\left(\frac{1}{4} + \frac{5}{7}\right) - \left(\frac{1}{4} - \frac{5}{7}\right)\right] = \frac{10}{7}$$

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Que 2 ⇒ If $\vec{f} = 3xy\,\hat{i} - y^2\hat{j}$, evaluate line integral $\int_C \vec{f}\cdot d\vec{r}$

where, c is the arc of parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$.

Solⁿ ⇒ We have to find line integral, i.e,

$$\int_C \vec{f}\cdot d\vec{r} = \int_C \vec{f}\cdot\left(\frac{d\vec{r}}{dt}\right) dt \longrightarrow (*)$$

First of all, we will find $\vec{r}$, i.e,

$\vec{r} = x\hat{i} + y\hat{j}$ (Here, integration is Performed in xy Plane, so $z=0$)

For this, we need to find x and y.
and, for x and y, we will take parametric form,
So, let $x = t$
then, $y = 2t^2$ ($\because y = 2x^2$ (given))

So, $\boxed{\vec{r} = t\hat{i} + 2t^2\hat{j}}$

$$\boxed{\frac{d\vec{r}}{dt} = \hat{i} + 4t\hat{j}} \longrightarrow (1)$$

and, $\vec{f} = 3xy\,\hat{i} - y^2\hat{j}$

$= 3(t)(2t^2)\,\hat{i} - (2t^2)^2\hat{j}$

$$\boxed{\vec{f} = 6t^3\,\hat{i} - 4t^4\hat{j}} \longrightarrow (2)$$

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Now, we want to find limits of integration, for this we will use given conditions,

At the point (0,0),

$x = 0 \Rightarrow t = 0. \quad (\because x = t)$

and at (1,2)

$x = 1 \Rightarrow t = 1 \quad (\because x = t)$

So, $\boxed{t \text{ varies from } 0 \text{ to } 1.}$ → ③

Now, using the conditions ①, ② and ③ in ⊛, we get

$$\int_C \vec{f} \cdot d\vec{r} = \int_0^1 (6t^3\hat{i} - 4t^4\hat{j})(\hat{i} + 4t\hat{j})\,dt$$

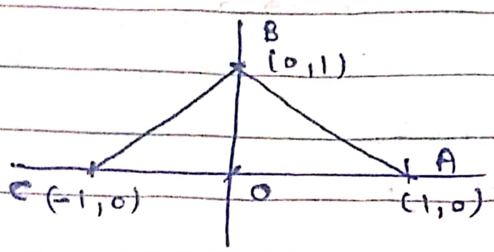
$$= \int_0^1 (6t^3 - 16t^5)\,dt$$

$$= \left[\frac{6t^4}{4} - \frac{16t^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \boxed{-\frac{7}{6}}$$

Que 3 → If $\vec{f} = y^2\hat{i} - x^2\hat{j}$, evaluate the integral $\int_C \vec{f} \cdot d\vec{r}$ about the triangle whose vertices are (1,0), (0,1) and (-1,0).

sol^n ⇒ Path of integration is the triangle ABC,



We want to find line integral, i.e,

$$\int_C \vec{f}\cdot d\vec{r} = \int_{AB} \vec{f}\cdot d\vec{r} + \int_{BC} \vec{f}\cdot d\vec{r} + \int_{CA} \vec{f}\cdot d\vec{r}$$

(∵ Integral about "C" represents the integral along the lengths of Δ)

Now, we need eqns. of lines AB, BC and CA,

① Eqn. of AB, A (1,0) B (0,1)

$$(y-y_1) = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$(y-0) = \frac{1-0}{0-1}(x-1)$$

$$\Rightarrow \boxed{y = 1-x}$$

$$\Rightarrow dy = -dx$$

Clearly, x varies from 1 to 0 on AB. →①

② Eqn. of BC,

$$y-1 = \frac{0-1}{-1-0}(x-0)$$

$\Rightarrow$ $\boxed{y = 1+x}$ x varies from 0 to -1.

$\Rightarrow$ $\boxed{dy = dx}$ $\longrightarrow$ ②

③ Eqn. of CA, (In a similar way)

$\boxed{y=0}$

so, $\boxed{dy=0}$ and x varies from -1 to 1.

$\longrightarrow$ ③

Now, $$\int_C \vec{f}\cdot\vec{dr} = \int_C (y^2\hat{i} - x^2\hat{j})\cdot(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C (y^2\,dx - x^2\,dy)$$

$$\int_C \vec{f}\cdot\vec{dr} = \int_{AB} (y^2dx - x^2dy) + \int_{BC} (y^2dx - x^2dy)$$

$$+ \int_{CA} (y^2dx - x^2dy)$$

$$= \int_1^0 [(1-x)^2dx - x^2(-dx)] + \int_0^{-1} (1+x)^2dx - x^2dx$$

$$+ \int_{-1}^{1} [(0)^2dx - x^2(0)]$$

(Using ①, ②, ③ cases)

So, $\int_C \vec{f} \cdot \vec{dr} = \int_1^0 (2x^2 - 2x + 1)\, dx + \int_0^{-1} (2x+1)\, dx$

$= \left[\frac{2x^3}{3} - x^2 + x\right]_1^0 + \left[x^2 + x\right]_0^{-1}$

$= \left[-\frac{2}{3} + 1 - 1\right] + \left[1 - 1\right]$

$= -\frac{2}{3}.$

Similarly, try other questions.